A NEW BOMBIERI-TYPE INEQUALITY

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HISTORY OF THE PROBLEM:

Theorem. Let P, Q be homogeneous polynomials of degrees m, n respectively. Then

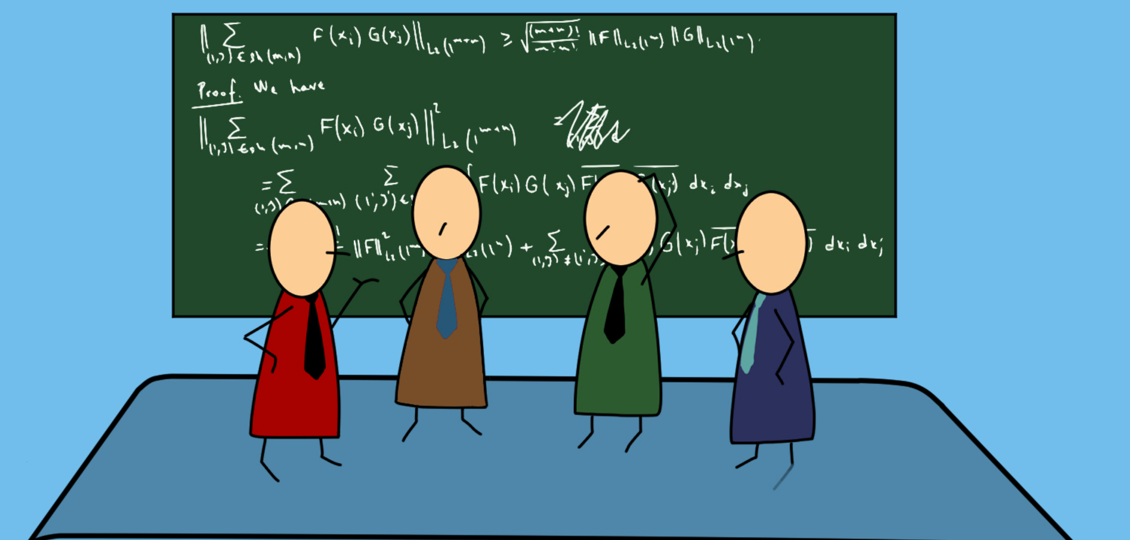
$$\|P\|\,\|Q\| \leq \sqrt{\frac{(m+n)!}{m!\,n!}}\,\|PQ\|$$

where $\|\cdot\|$ is the Bombieri norm and the inequality is sharp.

FAMOUS ARTICLE!

Product of polynomials in many variables, Journal of Number Theory 36, 2, 1990.

- B. Beauzamy
- E. Bombieri
- P. Enflo
- H. L. Montgomery



BOMBIERI NORM = KOSTLAN NORM = WEYL NORM

Given a bivariate homogeneous polynomial of degree N,

$$g(x,y) = \sum_{i=0}^{N} a_i x^i y^{N-i}, \quad a_i \in \mathbb{C}$$

where x, y are complex variables. The Bombieri norm of g satisfies

$$\|g\|^2 = \sum_{i=0}^{N} \binom{N}{i}^{-1} |a_i|^2 = \frac{N+1}{\pi} \int_{\mathbb{P}(\mathbb{C}^2)} \frac{|g(\eta)|^2}{\|\eta\|^{2N}}\, dV(\eta)$$

where the integration is made with respect to the volume form arising from the standard Riemannian structure in the complex projective space.

Every univariate degree N polynomial with complex coefficients $P(z) = \sum_{i=0}^{N} a_i z^i$ has a homogeneous counterpart $g(x,y) = \sum_{i=0}^{N} a_i x^i y^{N-i}$ and its Bombieri norm is defined via its homogenized version: $\|P\| = \|g\|$.

SOME PEOPLE WENT FURTHER

- P. B. Borwein — *Exact Inequalities for the Norms of Factors of Polynomials.*
- D. W. Boyd — *Sharp Inequalities for the Product of Polynomials.*
- D. Pinasco — *Lower bounds for norms of products of polynomials via bombieri inequality.*
- I.E. Pritsker and S. Ruscheweyh — *Inequalities for products of polynomials I, II.*

MAIN THEOREM

Given a set of complex points $z_1, \dots, z_N$ we have

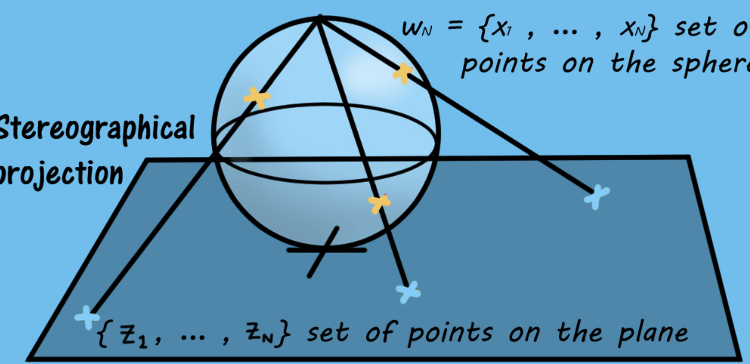
$$\prod_{i=1}^{N} \|x - z_i\| \leq \sqrt{\frac{e^N}{N+1}}\, \left\| \prod_{i=1}^{N} (x - z_i) \right\|$$

→ it's sharp!

→ The previously known best bound was given by the Bombieri inequality

$$\prod_{i=1}^{N} \|x - z_i\| \leq \sqrt{N!}\, \left\| \prod_{i=1}^{N} (x - z_i) \right\|.$$

Idea of the proof: go to the sphere



$P_N(x) = \prod_{i=1}^{N} (x - z_i)$ unique monic polynomial whose roots are $\{z_1, \dots, z_N\}$

The proof involves two mathematical objects

LOGARITHMIC ENERGY

The logarithmic energy of a set of points $\omega_N = \{x_1, \dots, x_N\}$ in the 2-dimensional sphere of radius 1 is defined as

$$\mathcal{E}_{\log}(\omega_N) = -\sum_{i \neq j} \log \|x_i - x_j\|.$$

The minimal value of the logarithmic energy and the configurations of points attaining such value are one of the main problems on Approximation Theory!

Theorem (Shub, Smale, 93'). If a set of points on the sphere has small logarithmic energy then the condition number of the associated polynomial is also small.

Theorem (Etayo, 19'). (A) If a polynomial $p(x) \in \mathbb{C}[x]$ has a small condition number, then the associated set of points on the sphere has small logarithmic energy.

CONDITION NUMBER

Given a polynomial $P(x)$ in $\mathbb{C}[x]$ and one of its roots z, the condition number of P at z is defined as

$$\mu(P, z) = \sqrt{N}\, \frac{\|P\| \,(1 + |z|^2)^{\frac{N}{2} - 1}}{|P'(z)|}$$

Note that the condition number quantifies how much the roots of a polynomial change when we perturbe a little bit the coefficients of the polynomial.

Armentano, Beltrán, Shub, 2011

$$\mathcal{E}_{\log}(\omega_N) = g(N) + N \log\left(\frac{\prod_{i=1}^{N} \sqrt{1 + |z_i|^2}}{\|P_N\|} \right) + \sum_{i=1}^{N} \log\left(\mu(P_N, z_i)\right), \quad \text{(B)}$$

where $g(N) = -\log(2) N^2 - \frac{N \log(N)}{2} + \log(2) N.$

A formula to rule them all

PROOF OF THE SHARPNESS:

1. We take a polynomial $p(x) \in \mathbb{C}[x]$ with optimal condition number. ← SOLVED RECENTLY BY BELTRÁN, ETAYO, MARZO & ORTEGA-CERDÀ!
2. Using Theorem (A) we obtain a bound for the logarithmic energy of the associated spherical points.
3. Through equation (B) we obtain an upper bound for $\frac{\prod_{i=1}^{N} \sqrt{1+|z_i|^2}}{\|P_N\|} = \frac{\prod_{i=1}^{N} \|x - z_i\|}{\left\| \prod_{i=1}^{N} (x - z_i) \right\|}$.

To prove the Main Theorem completely we need a few more analytical developments.

WANT TO KNOW MORE?

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